Mathematics 222B Lecture 14 Notes

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1 General Boundary Value Problems for Elliptic PDEs

1.1 How do we make sense of "regular" boundary value problems for eliiptic PDEs?

In this lecture, we will assume that P is an elliptic operator in divergence form:

$$Pu = -\partial_j(a^{j,k}\partial_k u) + b^j \partial_j u + cu.$$

Let U be an open, bounded, connected subset of $\mathbb{R}^d$ with C^1 boundary ∂U. A general boundary value problem might be of the form

$$\begin{cases} Pu = 0 & \text{in } U \\ Bu|_{\partial_U} = g & (\text{on } \partial U) \end{cases}$$

for some operator B.

So far, we have focused on the Dirichlet boundary condition

$$\begin{cases} Pu = 0 & \text{in } U \\ u|_{\partial_U} = g & (\text{on } \partial U) \end{cases}$$

By introducing an extension $\widetilde{g}$ of g to U, we could set, without loss of generality, $g = 0$. With this reduction, the problem we have considered is

$$\begin{cases} Pu = 0 & \text{in } U \\ u|_{\partial_U} = 0 & (\text{on } \partial U). \end{cases}$$

Our goal now is to generalize our elliptic theory to other boundary conditions. This will force us to consider what is a "regular" boundary value problem for PDEs. In order to solve a k-th order ODE, you need k pieces of data on the boundary. For the wave equation, which is a second order PDE, you impose boundary values and normal derivative values. Unlike ODEs, the wave equation, or Cauchy-Kovalevskaya, when we work with an elliptic PDE like $-\Delta u = f$, we do not prescribe the full $u, \frac{\partial}{\partial v} u$ on ∂U. How do we rigorously justify this high level discussion? We will see two approaches.

1.2 Weak formulations of boundary problems

Prove a uniqueness theorem via the energy method.

Example 1.1. If $P = -\Delta$ and we are solving

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ u|_{\partial_U} = g & (\text{on } \partial U) \end{cases}$$

then

$$0 = \int_U -\Delta u u \, dx = \int |Du|^2 \, dx.$$

Note the parallel between this basic consideration and our weak formulation of the Dirichlet problem: $u \in H^1$ solves the Dirichlet problem

$$\begin{cases} Pu = f & \text{in } U \\ u|_{\partial_U} = g & (\text{on } \partial U) \end{cases}$$

if and only if $u \in H_0^1(U)$ and $-\Delta u = f$ in the sense of $\mathcal{D}'(U)$. This is equivalent to

$$\int_U a^{j,k} \partial u \partial_k \varphi + b^j \partial_j u \varphi + cu\varphi \, dx = \int_U f\varphi \, dx \qquad \forall \varphi \in H_0^1(U).$$

We will try to generalize this weak formulation to other boundary conditions.

Example 1.2. Consider the Neumann boundary condition

$$\begin{cases} Pu = f & \text{in } U \\ \nu^j \partial_j u|_{\partial_U} = g & (\text{on } \partial U) \end{cases}$$

We can rewrite this as

$$\begin{cases} Pu = f & \text{in } U \\ a^{j,k} \nu_k \partial_j u|_{\partial_U} = g & (\text{on } \partial U) \end{cases}$$

In the case of the Laplace equation, this is the same. From the point of view of differential geometry, this is a more natural quantity to look at because ν_k is dh, where h is the boundary defining form. The natural Riemannian metric in this problem is a. By an extension procedure, we can write the problem as

$$\begin{cases} Pu = f & \text{in } U \\ a^{j,k} \nu_k \partial_j u|_{\partial_U} = 0 & (\text{on } \partial U) \end{cases}$$

For simplicity, assume $b = c = 0$. Then we have the formal computation

$$\int_U f\varphi \, dx = \int_U -\partial_j (a^{j,k} \partial_j u) \varphi \, dx = \int_U a^{j,k} \partial_j u \partial_k \varphi \, dx - \int_{\partial U} \underbrace{\nu_j a^{j,k} \partial_k u}_{=0} \varphi \, dA.$$

This motivates the following definition:

Definition 1.1. We say that u **satisfies the Neumann boundary problem** if for all $\varphi \in H^1(U)$,

$$\int_U a^{j,k}\partial_j u \partial_k \varphi \, dx = \int_U f\varphi \, dx.$$

Remark 1.1. If $u \in C^1$ then this formulation should be equivalent to the classical one.

Once we formulate the problem like this, the L^2 theory is easy to generalize.

Theorem 1.1. *Suppose ∂U is C^1, $a \succ \lambda I$ in U, and $a \in L^\infty$ (also $b, c \in L^\infty$). Then*

1. *For any $\mu \in \mathbb{R}$, the map $u \mapsto Pu - \mu u$ associated to the Neumann boundary value problem*

$$(\mathrm{NP}_\mu) \begin{cases} Pu - \mu u = f & \text{in } U \\ a^{j,k}\nu_j \partial_j u|_{\partial_U} = g & \text{(on } \partial U\text{)} \end{cases}$$

is Fredholm with index 0 from $H^1(U) \to (H^1(U))^* \subseteq H^{-1}(U)$. That is, one of the following holds:

   (i) For all $f \in L^2(U)$, there exists a unique $u \in H^1$ which solves the Neumann boundary problem (NP_μ).

   (ii) There exists a solution $v \neq 0$ to (NP_μ) with $f = 0$. Furthermore, for $\mu \gg 1$, alternative (i) applies.

2. *If ∂U is C^k and $a, b, c \in C^k$, then*

$$\|u\|_{H^{1+k}(U)} \lesssim \|f\|_{H^{k-1}(U)} + \|u\|_{H^k(U)}.$$

Example 1.3. Take $P = -\Delta$ and solve

$$\begin{cases} -\Delta u = 0 \\ u|_{\partial U} = 0. \end{cases}$$

This has a nontrivial solution $v = \text{const} \neq 0$.

This leads to solvability for f orthogonal to the kernel of the adjoint. In this case, this is equivalent to $\int_U f \, dx = 0$.

For other boundary conditions, this weak formulation also makes sense.

Definition 1.2. We say that u **satisfies the Robin boundary problem** if for all $\varphi \in H^1(U)$,

$$\int_U a^{j,k}\partial_j u \partial_k \varphi \, dx + \int_{\partial U} \alpha u \varphi \, dS = \int_U f\varphi \, dx.$$

Example 1.4 (Oblique Dirichlet boundary condition)**.** Suppose $b = c = 0$, and consider the problem

$$(\mathrm{OP}) \begin{cases} Pu = f \\ X^j \partial_j u = 0, \end{cases}$$

where X is transversal to ∂U, outward. Then $X = X^\perp + X^\top$, where $X^\perp$ is parallel to $a^{j,k}\nu_k \vec{e_k}$. Normalize to make $X^\perp = a^{j,k}\nu_j \vec{e_k}$. This tells us that

$$\int_U a^{j,k}\partial_j u \partial_k \varphi + \int_{\partial U} X^\top u\varphi \, dA = \int_U f\varphi \, dx.$$

The second term is trickier to make sense of, since we need to make sense of the trace. As an exercise, check that $\int_{\mathbb{R}^{d-1}} \partial uv \, dx$ is well defined for $u, v \in H^{1/2}(\mathbb{R}^{d-1})$. This is just barely well-defined, however, in the sense of the trace theorem needing $H^{1/2}$.

1.3 The "microlocal" formulation

The reference for this section is volume 1 of Taylor's PDE book, section 5.11. Look at the Laplace equation $-\Delta u = 0$ in the half space $\mathbb{R}^d_+$. Write z for the last variable and x for the remaining $d-1$ variables, so this is $-\partial_z^2 - \Delta_x u = 0$. Suppose we have boundary conditions $u|_{\partial U} =?$ and $\partial_z u|_{\partial U} =?$. We can view this as an evolution equation in the z variable and take the Fourier transform in x to get

$$(-\partial_z^2 + |\xi|^2)\widehat{u} = 0$$

with boundary conditions $\widehat{u}|_{z=0} = g$ and $\partial_z \widehat{u}|_{z=0} = h$. This gives

$$\widehat{(}z, \xi) = a_+(\xi)e^{|\xi|z} + a_-(\xi)e^{-|\xi|z}.$$

However, the first term $e^{|\xi|z}$ is a problem because growth in Fourier space corresponds to a lack of regularity in physical space. So in order to have boundary regularity, we want $a_+(\xi) = 0$. This means that we are only left with half of the full freedom to choose $\widehat{g}$ and $\widehat{h}$.

The claim is that the constant coefficient picture generalizes to the variable coefficient picture. The idea is that using the technique of "freezing the coefficients," we can formulate the notion of a "regular" elliptic boundary value problem, for which we have elliptic regularity and the Fredholm property, based on the constant coefficient computation.

Here, we assume that $a, b, c \in C^\infty(\overline{U})$ and that ∂U is C^∞.

Definition 1.3. For $k \geq 1$, define

$$H^{k-1/2}(\partial U) = \{g = v|_{\partial U} : v \in H^k(U)\},$$

with the norm

$$\|g\|_{H^{k-1/2}(\partial U)} = \inf_{u : u|_{\partial U} = g} \|u\|_{H^k(U)}.$$

Remark 1.2. If we define fractional Sobolev spaces on manifolds, this will actually be the $k - 1/2$ Sobolev space on ∂U.

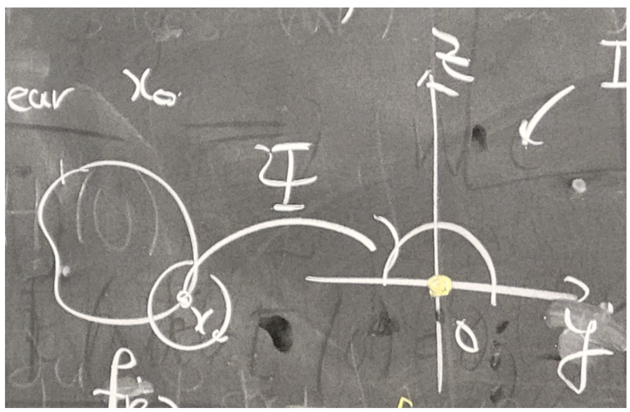
Now consider the boundary problem

$$\begin{cases} Pu = f & \text{in } U \\ Bu|_{\partial U} = g. \end{cases}$$

Here, we assume that $P : C^\infty(U) \to C^\infty(U)$ and $B(\cdot)|_{\partial U} : C^\infty(U) \to C^\infty(\partial U)$. Given $x_0 \in \partial U$, there exists a boundary straightening map near x_0. In these variables, write

$$P = -\partial_z^2 + P_1(y, z, D_y^2)\,\partial_z + P_0(y, z, D_y, D_y^2),$$

$$B = b\partial_z + B_0(y, z, \partial_y).$$



Say x_0 is mapped to 0, and let P_{x_0} be the frozen constant coefficient operator

$$P_{x_0} = -\partial_z^2 + P_1(0, 0, D_y)\partial_z + P_0(0, 0, ; D_y, D_y^2),$$

$$B_{x_0} = b(0, 0)\partial_z + B_0(0, 0, \partial_y).$$

Definition 1.4. A boundary value problem is a **regular elliptic boundary value problem** if for all $x_0 \in \partial U$, for all $\xi \in \mathbb{R}^{d-1}$, and for all ζ, there exists a unique bounded solution to the ODE

$$P_{x_0}\widehat{u}(z, \xi) = 0, \qquad B_{x_0}\widehat{u}(z, \xi) = \zeta.$$

This is called the **Loputinski-Shapiro condition**. This is like if we pretend we take the Fourier transform and replace ∂_y by $c\xi$. This condition gives an ODE in z.

Theorem 1.2. *For a regular elliptic boundary value problem, the map* $H^{k+2}(U) \ni u \mapsto (Pu, Bu) \in H^k(U) \times H^{k-(\text{order } B)-1/2}(\partial U)$ *is Fredholm, and we have elliptic (boundary) regularity*

$$\|u\|_{H^{k+2}(U)} \lesssim \|f\|_{H^k(U)} + \|Bu\|_{H^{k-(\text{order } B)-1/2}(\partial U)} + \|u\|_{H^{k+1}(U)}.$$